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THE READJUSTMENT OF CERTAIN UNSTABLE ATMOSPHERIC SYSTEMS UNDER CONSERVATION OF VORTICITY [1]

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In his memoir "On the Energy of Storms", Margules devised methods for the calculation of the maximum kinetic energy of atmospheric systems that can result from the rearrangement of two air masses of differing properties, from an initial unstable arrangement to one possessing stability. Although his results show that the kinetic energy made available by these processes is sufficient to account for observed wind velocities, no attempt was made by Margules to deduce the probable distribution of this kinetic energy within the air masses which comprise the systems studied. In order to secure some idea of the character that this distribution might be expected to assume, a problem similar in certain respects to

earth's rotation on all the motions involved in the readjustment? Also, what will be the distribution of velocities generated by the action of pressure gradients and of Coriolis forces within each mass of fluid?

In outline the changes that take place upon the transition to a state of equilibrium may be described as follows: A portion of the denser fluid located in the region A in the initial state will tend to be displaced to the right. Since this displacement is accompanied by a Coriolis force acting at right angles to the direction of motion, a transverse horizontal velocity will be generated in this portion of fluid. In a steady state this velocity must be accompanied by a gradient of pressure which in this case is supplied by a
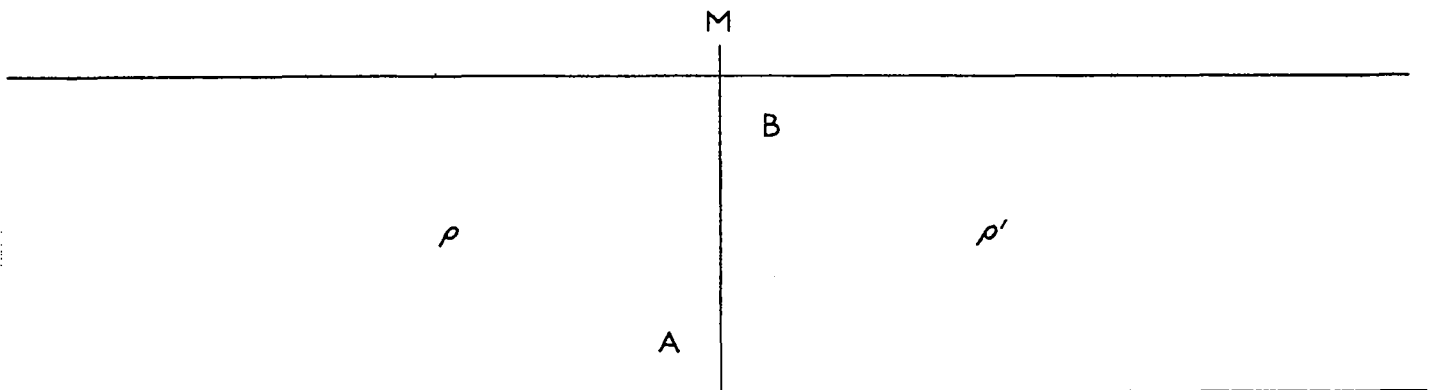

FIGURE 1.—Two fluid masses in the initial state.

some of the cases treated by Margules will be formulated and solved making use of a principle which may be alluded to as that of conservation of absolute vorticity. Because of the great complexities which arise in an attempt to deal with a compressible atmosphere, attention will be confined to a system composed of two homogeneous and incompressible fluids of slightly differing densities. It will also be assumed in the treatment that all frictional effects are absent.

Let it be supposed that in the initial state two masses of fluid of differing densities lie side by side, at rest relative to the earth, separated by a vertical plane as in figure 1, with the lighter fluid to the right. The horizontal layer thus formed is considered as being of uniform thickness and extending an infinite distance to the right and to the left. The problem for which a solution is sought may be stated as follows: Assuming no mixing of the fluids, what will be the shape of the free surface and of the internal boundary, if the system is allowed to come infinitely slowly to equilibrium, taking into consideration the effect of the

mutual adjustment of the slope of the free surface and of the internal boundary separating the liquids. Likewise a portion of the lighter fluid originally situated at B will tend to move to the left, bringing about a transverse velocity in the opposite sense. The requisite pressure gradient in this case is produced by the deformation of the free surface alone.

Due to the fact that continuity of depth must be preserved throughout the layer, it must be assumed that some lateral movement with resulting transverse velocities must occur in the regions farther to the left and to the right, decreasing to zero at great distances from the original discontinuity. It is now possible to schematically represent the final state as in figure 2.

It thus appears that for the purpose of analysis the system may be divided into three regions, regions I and III in figure 2 consisting of a single layer, while region II consists of two superimposed layers. The distances c and a are to be obtained as a result of the investigation. In the above diagram the positive y direction is taken to the left and the positive x direction as perpendicular to the plane of the figure away from the reader, while the origin is located

at O. For convenience, the diagram is taken in the plane of a meridian with the positive y direction extending northward. In the Northern Hemisphere this leads to easterly winds in the regions marked E and westerly winds in the region marked W. Since the equations of motion used in meteorology are independent of the azimuthal orientation of the axes, this choice of coordinates will not produce any lack of generality in the results.

Restricting our attention for the present to region I, let it be assumed that a column of fluid which in the initial state was located at a point $y_0{}^I$ and had a width $dy_0{}^I$ and a height D_0 is now located at a point y, has a width dy and a height D^I. The principle of continuity of mass gives

In the above p_g may be taken as the pressure at the surface, for which the value $g\rho D^I$ may be substituted. Substituting also for u^I from (5),

$$0 = -f^2(y - y_0{}^I) - \frac{g\,dD^I}{dy}. \tag{7}$$

Substituting for $\dfrac{dD^I}{dy}$ from (3),

$$0 = (y - y_0{}^I) + \frac{gD_0}{f^2}\frac{d^2y_0{}^I}{dy^2}. \tag{8}$$

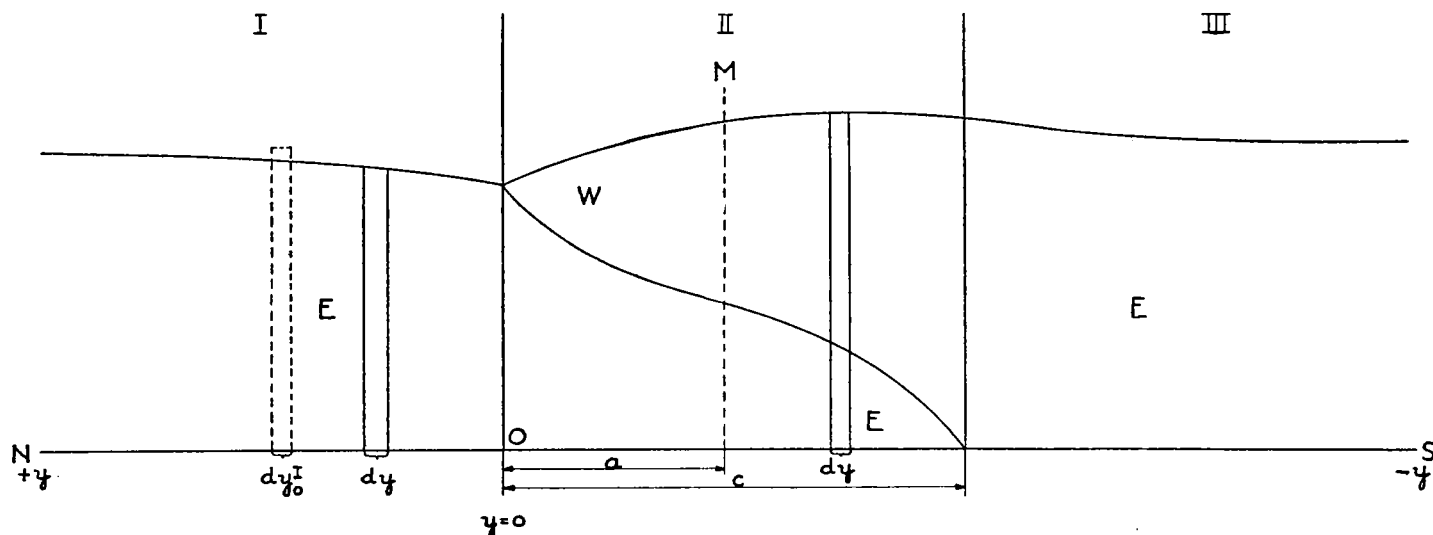Since the quantities g, D_0 and f are considered constant in

FIGURE 2.—Schematic diagram of the final state.

$$\rho D_0 dy_0{}^I = \rho D^I dy \tag{1}$$

or

$$D^I = D_0 \frac{dy_0{}^I}{dy} \tag{2}$$

where g is the acceleration of gravity and ρ is the density. By differentiation,

$$\frac{dD^I}{dy} = D_0 \frac{d^2y_0{}^I}{dy^2} \tag{3}$$

Indicating velocities in the x direction by u^I and velocities in the y direction by v the equation of motion for the transverse velocity of the element, since no net external forces are present, is

$$\frac{du^I}{dt} = fv = f\frac{dy}{dt}, \tag{4}$$

where f is the Coriolis parameter $2\,\Omega\,\sin\phi$. Integrating between the limits $y_0{}^I$ and y

$$u^I = f(y - y_0{}^I). \tag{5}$$

This implies that the velocities are constant with elevation and therefore the pressure gradients must be independent of elevation—a result which agrees with hydrostatical considerations.

Since a steady state is assumed, in the final condition the gradient wind equation must be fulfilled. For straight line flow, this will be

$$0 = -fu^I - \frac{1}{\rho}\frac{\partial p_g}{\partial y}. \tag{6}$$

this whole discussion and since the factor $\dfrac{gD_0}{f^2}$ recurs frequently, a quantity λ having the dimensions of a length will be defined as follows, and substituted in the equations:

$$\lambda = \frac{\sqrt{gD_0}}{f} \tag{9}$$

Equation (8) now becomes,

$$\frac{d^2y_0{}^I}{dy^2} + \frac{1}{\lambda^2}(y - y_0{}^I) = 0. \tag{10}$$

The solution of this equation is easily shown to be

$$y_0{}^I = y + Q^I e^{-\frac{y}{\lambda}} + Q^{III} e^{\frac{y}{\lambda}} \tag{11}$$

In this equation Q^I and Q^{III} are constants of integration which must be fixed by boundary conditions while e is the Napierian base. Obviously Q^{III} must be zero in region I since the liquid is to be undisturbed at great distances to the left. This is only possible if $y^I{}_0 = y$ for large values of y. The value of Q^I on the other hand cannot be determined at this point, but must be evaluated later, simultaneously with the determination of other similar integration constants for the remaining two regions. The fundamental equation for region I, giving the lateral change of position of a particle from the initial to the final state may be written,

$$y_0{}^I = y + Q^I e^{-\frac{y}{\lambda}} \tag{12}$$

From this equation the velocity distribution may be obtained by the aid of equation (5) and is found to be,

$$u^{\mathrm{I}} = -fQ^{\mathrm{I}} e^{-\frac{y}{\lambda}} \tag{13}$$

By differentiating (12) with respect to y and combining the result with (2) an expression for the shape of the free surface results,

$$D^{\mathrm{I}} = D_0 \left[1 - \frac{1}{\lambda} Q^{\mathrm{I}} e^{-\frac{y}{\lambda}} \right] \tag{14}$$

Considering now the conditions in region III, an exactly analogous line of reasoning may be followed as was used above, and an equation comparable to (11) may be set down,

$$y_0^{\mathrm{III}} = y + Q^{\mathrm{I}} e^{-\frac{y}{\lambda}} + Q^{\mathrm{III}} e^{\frac{y}{\lambda}}. \tag{15}$$

It is clear, however, that now the constant Q^{I} must be zero since the fluid is not disturbed at great distances to the right, and the fundamental equation for region III becomes

$$y_0^{\mathrm{III}} = y + Q^{\mathrm{III}} e^{\frac{y}{\lambda}}. \tag{16}$$

The constant Q^{III} will be evaluated later.

Expressions for the velocity distribution and for the shape of the free surface may be written immediately as in the case of region I. These are respectively,

$$u^{\mathrm{III}} = -fQ^{\mathrm{III}} e^{\frac{y}{\lambda}}, \tag{17}$$

and

$$D^{\mathrm{III}} = D_0 \left[1 + \frac{1}{\lambda} Q^{\mathrm{III}} e^{\frac{y}{\lambda}} \right]. \tag{18}$$

Turning our attention to region II, the situation is more complicated due to the presence of a double layer. Retaining the convention of indicating quantities referring to the initial state by the subscript zero, and designating quantities applying to the upper layer by primes, an elemental vertical column extending from the surface to the top will now be considered. Corresponding to equation (1), which specified the requirement of continuity of mass, there are now two such equations:

$$\rho' D_0' dy_0^{\mathrm{II}'} = \rho' D^{\mathrm{II}'} dy, \tag{19}$$

for the upper layer and

$$\rho D_0 dy_0^{\mathrm{II}} = \rho D^{\mathrm{II}} dy \tag{20}$$

for the lower layer. In these two equations a distinction is made between the quantities D_0 and D_0'. Although according to the statement of the problem made previously these are equal, this simplification will not be made at this point, in view of the possible application of the equations derived to more general problems than the one dealt with at present. Solving (19) and (20) for the depths in the final state,

$$D^{\mathrm{II}'} = D_0' \frac{dy_0^{\mathrm{II}'}}{dy}, \tag{21}$$

and

$$D^{\mathrm{II}} = D_0 \frac{dy_0^{\mathrm{II}}}{dy} \tag{22}$$

Differentiating the above with respect to y

$$\frac{dD^{\mathrm{II}'}}{dy} = D_0' \frac{d^2 y_0^{\mathrm{II}'}}{dy^2}, \tag{23}$$

and

$$\frac{dD^{\mathrm{II}}}{dy} = D_0 \frac{dy_0^{\mathrm{II}}}{dy^2}. \tag{24}$$

Corresponding to equation (5) there are two expressions for velocity, namely

$$u^{\mathrm{II}'} = f(y - y_0^{\mathrm{II}'}), \tag{25}$$

and

$$u^{\mathrm{II}} = f(y - y_0^{\mathrm{II}}). \tag{26}$$

We also have two gradient wind equations:

$$0 = -fu^{\mathrm{II}'} - \frac{1}{\rho'} \frac{\partial p_g'}{\partial y}, \tag{27}$$

for the upper layer and

$$0 = -fu^{\mathrm{II}} - \frac{1}{\rho} \frac{\partial p_g}{\partial y} \tag{28}$$

for the lower. A question now arises as to the interpretation of p_g' in equation (27). Since (27) involves only the gradient of pressure and not the absolute value of the pressure, the derivative appearing may be evaluated by forming an expression for the pressure distribution along any level surface in the upper layer, and differentiating. This can be done conveniently by considering the pressure at a level surface situated at an arbitrary elevation Z. Thus

$$p_g' = g\rho' (D^{\mathrm{II}'} + D^{\mathrm{II}} - Z). \tag{29}$$

Differentiating and changing to ordinary derivatives,

$$\frac{dp_g'}{dy} = g\rho' \left(\frac{dD^{\mathrm{II}'}}{dy} + \frac{dD^{\mathrm{II}}}{dy} \right). \tag{30}$$

For the lower layer,

$$p_g = g\rho' D^{\mathrm{II}'} + g\rho D^{\mathrm{II}} \tag{31}$$

and

$$\frac{dp_g}{dy} = g\rho' \frac{dD^{\mathrm{II}'}}{dy} + \frac{g\rho dD^{\mathrm{II}}}{dy}. \tag{32}$$

Substituting in equations (27) and (28) from (25) and (26) and also from (30) and (32)

$$0 = -f^2(y - y_0^{\mathrm{II}'}) - g\left[\frac{dD^{\mathrm{II}'}}{dy} + \frac{dD^{\mathrm{II}}}{dy} \right], \tag{33}$$

and

$$0 = -f^2(y - y_0^{\mathrm{II}}) - \frac{g}{\rho}\left[\rho' \frac{dD^{\mathrm{II}'}}{dy} + \frac{\rho dD^{\mathrm{II}}}{dy} \right] \tag{34}$$

Substituting in the above two equations for the derivatives of the depths the values given by (23) and (24), we get:

$$0 = -f^2(y - y_0^{\mathrm{II}'}) - g\left[D_0' \frac{d^2 y_0^{\mathrm{II}'}}{dy^2} + D_0 \frac{d^2 y_0^{\mathrm{II}}}{dy^2} \right] \tag{35}$$

and

$$0 = -f^2(y - y_0^{\mathrm{II}}) - g\left[D_0' \frac{\rho'}{\rho} \frac{d^2 y_0^{\mathrm{II}'}}{dy^2} + D_0 \frac{d^2 y_0^{\mathrm{II}}}{dy^2} \right] \tag{36}$$

We thus finally have a pair of second order simultaneous differential equations involving the single independent

variable y and two dependent variables $y_0^{II'}$ and y_0^{II}, which must be solved. Customary methods of separating the dependent variables lead to two fourth order equations whose solutions contain eight constants of integration. Rather than to pursue this course the separation of variables will be effected as follows: [1] Let the first equation be multiplied by an undetermined constant factor α, so that the equations have the form,

$$0=-f^2(\alpha y-\alpha y_0^{II'})-g\left[\alpha D_0'\frac{d^2 y_0^{II'}}{dy^2}+\alpha D_0\frac{d^2 y_0^{II}}{dy^2}\right]. \quad (37)$$

$$0=-f^2(y-y_0^{II})-g\left[D_0'\frac{\rho'}{\rho}\frac{d^2 y_0^{II'}}{dy^2}+D_0\frac{d^2 y_0^{II}}{dy^2}\right]. \quad (38)$$

Adding,

$$0=f^2[y_0^{II}+\alpha y_0^{II'}-(1+\alpha)y]$$
$$-g\left[\left(\frac{\rho'}{\rho}+\alpha\right)D_0'\frac{d^2 y_0^{II'}}{dy^2}+(1+\alpha)D_0\frac{d^2 y_0^{II}}{dy}\right]. \quad (39)$$

Rearranging,

$$0=f^2[y_0^{II}+\alpha y_0^{II'}-(1+\alpha)y]$$
$$-gD_0(1+\alpha)\frac{d^2}{dy^2}\left[y_0^{II}+\frac{\frac{\rho'}{\rho}+\alpha}{1+\alpha}\frac{D_0'}{D_0}y_0^{II'}\right]. \quad (40)$$

Let the value of α be now defined so as to make the coefficient of $y_0^{II'}$ in the first brackets equal to the coefficient of $y_0^{II'}$ in the second brackets. This gives the relation

$$\alpha=\frac{\rho'+\rho\alpha}{\rho(1+\alpha)}\frac{D_0'}{D_0}, \quad (41)$$

which has two roots,

$$\alpha_1=-\frac{D_0-D_0'}{2D_0}+\frac{1}{2}\sqrt{\frac{(D_0-D_0')^2}{D_0^2}+\frac{4\rho'D_0'}{\rho D_0}} \quad (42)$$

and

$$\alpha_2=-\frac{D_0-D_0'}{2D_0}-\frac{1}{2}\sqrt{\frac{(D_0-D_0')^2}{D_0^2}+\frac{4\rho'D_0'}{\rho D_0}}. \quad (43)$$

Let two dependent variables A_1 and A_2 be defined by the following:

$$A_1=y_0^{II}+\alpha_1 y_0^{II'}, \quad (44)$$

$$A_2=y_0^{II}+\alpha_2 y_0^{II'}. \quad (45)$$

Introducing these into (40) two equations in which the variables are separated are obtained:

$$0=A_1-(1+\alpha_1)y-\frac{gD_0}{f^2}(1+\alpha_1)\frac{d^2 A_1}{dy^2}; \quad (46)$$

$$0=A_2-(1+\alpha_2)y-\frac{gD_0}{f^2}(1+\alpha_2)\frac{d^2 A_2}{dy^2}. \quad (47)$$

Introducing λ and rearranging,

$$\frac{d^2 A_1}{dy^2}-\frac{1}{(1+\alpha_1)\lambda^2}[A_1-(1+\alpha_1)y]=0; \quad (48)$$

$$\frac{d^2 A_2}{dy^2}-\frac{1}{(1+\alpha_2)\lambda^2}[A_2-(1+\alpha_2)y]=0. \quad (49)$$

[1] This procedure was suggested to the writer by Prof. C. G. Rossby.

The solutions of (48) and (49) may be set down immediately in the form

$$A_1=(1+\alpha_1)y+K_1 e^{\kappa_1 y}+Q_1 e^{-\kappa_1 y} \quad (50)$$

and

$$A_2=(1+\alpha_2)y+K_2 e^{\kappa_2 y}+Q_2 e^{-\kappa_2 y}, \quad (51)$$

in which K_1, K_2, Q_1, and Q_2 are constants of integration and κ_1 and κ_2 are given by

$$\kappa_1=\frac{1}{\lambda\sqrt{1+\alpha_1}}; \quad (52)$$

$$\kappa_2=\frac{1}{\lambda\sqrt{1+\alpha_2}} \quad (53)$$

Reverting to the original dependent variables, $y_0^{II'}$ and y_0^{II}, we have from (44) and (45)

$$A_1-A_2=\alpha_1 y_0^{II'}-\alpha_2 y_0^{II'}=y_0^{II'}(\alpha_1-\alpha_2), \quad (54)$$

and

$$y_0^{II'}=\frac{A_1-A_2}{\alpha_1-\alpha_2}. \quad (55)$$

Similarly,

$$\alpha_2 A_1-\alpha_1 A_2=\alpha_2 y_0^{II}-\alpha_1 y_0^{II}=y_0^{II}(\alpha_2-\alpha_1), \quad (56)$$

and

$$y_0^{II}=\frac{\alpha_2 A_1-\alpha_1 A_2}{\alpha_2-\alpha_1}. \quad (57)$$

Substituting in (55) and (57) from (50) and (51),

$$y_0^{II'}=y+\frac{1}{\alpha_1-\alpha_2}\left[K_1 e^{\kappa_1 y}-K_2 e^{\kappa_2 y}+Q_1 e^{-\kappa_1 y}-Q_2 e^{-\kappa_2 y}\right]; \quad (58)$$

and

$$y_0^{II}=y+\frac{1}{\alpha_2-\alpha_1}[\alpha_2 K_1 e^{\kappa_1 y}-\alpha_1 K_2 e^{\kappa_2 y}+\alpha_2 Q_1 e^{-\kappa_1 y}-\alpha_1 Q_2 e^{-\kappa_2 y}]. \quad (59)$$

Let the simplification now be made that $D'_0=D_0$. From (42) and (43) it is seen that

$$\alpha_1=+\sqrt{\frac{\rho'}{\rho}}; \quad (60)$$

$$\alpha_2=-\sqrt{\frac{\rho'}{\rho}}. \quad (61)$$

In other words we may say that

$$\alpha_1=-\alpha_2=\alpha=\sqrt{\frac{\rho'}{\rho}}. \quad (62)$$

The final form of the fundamental equations for region II thus becomes,

$$y_0^{II'}=y+\frac{1}{2\alpha}[K_1 e^{\kappa_1 y}-K_2 e^{\kappa_2 y}+Q_1 e^{-\kappa_1 y}-Q_2 e^{-\kappa_2 y}], \quad (63)$$

$$y_0^{II}=y+\frac{1}{2}[K_1 e^{\kappa_1 y}+K_2 e^{\kappa_2 y}+Q_1 e^{-\kappa_1 y}+Q_2 e^{-\kappa_2 y}]. \quad (64)$$

By the aid of (25) and (26) expressions for the velocities in the two layers are obtained:

$$u^{II'}=-\frac{f}{2\alpha}[K_1 e^{\kappa_1 y}-K_2 e^{\kappa_2 y}+Q_1 e^{-\kappa_1 y}-Q_2 e^{-\kappa_2 y}], \quad (65)$$

$$u^{II}=-\frac{f}{2}[K_1 e^{\kappa_1 y}+K_2 e^{\kappa_2 y}+Q_1 e^{-\kappa_1 y}+Q_2 e^{-\kappa_2 y}]. \quad (66)$$

By means of (22) and (64) the position of the internal boundary may be specified:

$$D^{II}=D_0\left[1+\frac{1}{2}(K_1\kappa_1e^{\kappa_1 y}+K_2\kappa_2e^{\kappa_2 y}-Q_1\kappa_1e^{-\kappa_1 y}-Q_2\kappa_2e^{-\kappa_2 y})\right].\quad(67)$$

The thickness of the upper layer can be obtained in a similar manner from (21) and (63):

$$D^{II\prime}=D_0\left[1+\frac{1}{2\alpha}(K_1\kappa_1e^{\kappa_1 y}-K_2\kappa_2e^{\kappa_2 y}-Q_1\kappa_1e^{-\kappa_1 y}+Q_2\kappa_2e^{-\kappa_3 y})\right].\quad(68)$$

The shape of the free surface is given by the sum of $D^{II\prime}$ and D^{II} for each value of y.

Having derived the fundamental equations (12) and (16) for regions I and III, respectively, and (63) and (64) for region III, it is next necessary to determine simultaneously the six integration constants Q^I, Q^{III}, K_1, K_2, Q_1, and Q_2 and also the constants c and a. In order to accomplish this it is necessary to have eight equations not involving the independent and dependent variables. By examining the requirements at the two vertical boundaries separating the three regions it will be seen that there exist exactly eight independent boundary conditions which can be interpreted in the form of the eight required equations.

Taking first the vertical at $y=0$ in figure 2, the fluid in the upper layer of region II at this point originally was located at the discontinuity, hence the condition exists that when

$$\left.\begin{array}{l}y=0\\y_0{}^{II\prime}=-a\end{array}\right\}.\quad(69)$$

Also it was assumed that at this point the thickness of the upper layer of region II becomes zero, hence according to (21) when

$$\left.\begin{array}{l}y=0\\\dfrac{dy_0{}^{II\prime}}{dy}=0\end{array}\right\}.\quad(70)$$

Since the fluid in region I is in contact with the fluid in the lower layer of region II at $y=0$, there must be continuity in the displacements at this point so that

$$\left.\begin{array}{l}y=0\\y_0{}^I=y_0{}^{II}\end{array}\right\}.\quad(71)$$

Finally, since the depth of the fluid in the lower layer of region II must be the same as that of region I at this point, we have from (2) and (22)

$$\left.\begin{array}{l}y=0\\\dfrac{dy_0{}^I}{dy}=\dfrac{dy_0{}^{II}}{dy}\end{array}\right\}.\quad(72)$$

At the vertical boundary between regions II and III there exist four additional conditions which are comparable to the four already given. Expressing the displacement of the lower layer of region II at this point, we have the condition

$$\left.\begin{array}{l}y=-c\\y_0{}^{II}=-a\end{array}\right\}.\quad(73)$$

The vanishing of the thickness of the lower layer gives the condition that

$$\left.\begin{array}{l}y=-c\\\dfrac{dy_0{}^{II}}{dy}=0\end{array}\right\}.\quad(74)$$

Continuity of displacements gives the relation that when

$$\left.\begin{array}{l}y=-c\\y_0{}^{III}=y_0{}^{II\prime}\end{array}\right\}.\quad(75)$$

Finally again continuity of depth gives the condition

$$\left.\begin{array}{l}y=-c\\\dfrac{dy_0{}^{III}}{dy}=\dfrac{dy_0{}^{II\prime}}{dy}\end{array}\right\}.\quad(76)$$

Imposing these eight boundary conditions in the order given upon the fundamental equations (12), (16), (63), and (64), there result the eight equations written below:

$$-a\alpha=\frac{1}{2}[K_1-K_2+Q_1-Q_2]\quad(77)$$

$$-\alpha=\frac{1}{2}[K_1\kappa_1-K_2\kappa_2-Q_1\kappa_1+Q_2\kappa_2]\quad(78)$$

$$Q^I=\frac{1}{2}[K_1+K_2+Q_1+Q_2]\quad(79)$$

$$-\frac{1}{\lambda}Q^I=\frac{1}{2}[K_1\kappa_1+K_2\kappa_2-Q_1\kappa_1-Q_2\kappa_2]\quad(80)$$

$$c-a=\frac{1}{2}[K_1e^{-\kappa_1 c}+K_2e^{-\kappa_2 c}+Q_1e^{\kappa_1 c}+Q_2e^{\kappa_2 c}]\quad(81)$$

$$-1=\frac{1}{2}[K_1\kappa_1e^{-\kappa_1 c}+K_2\kappa_2e^{-\kappa_2 c}-Q_1\kappa_1e^{\kappa_1 c}-Q_2\kappa_2e^{\kappa_2 c}]\quad(82)$$

$$\alpha Q^{III}e^{-\frac{c}{\lambda}}=\frac{1}{2}[K_1e^{-\kappa_1 c}-K_2e^{-\kappa_2 c}+Q_1e^{\kappa_1 c}-Q_2e^{\kappa_2 c}]\quad(83)$$

$$\alpha Q^{III}\frac{1}{\lambda}e^{-\frac{c}{\lambda}}=\frac{1}{2}[K_1\kappa_1e^{-\kappa_1 c}-K_2\kappa_2e^{-\kappa_2 c}-Q_1\kappa_1e^{\kappa_1 c}+Q_2\kappa_2e^{\kappa_2 c}]\quad(84)$$

It will be noted that all the constants enter these equations linearly except c, which is contained both linearly and exponentially. It is therefore possible to eliminate the six constants of integration and a algebraically, the result being a single exponential equation in c which can be solved by trial for any particular numerical example. When this has been accomplished, the remaining constants may be determined without difficulty. The actual process of carrying out the elimination becomes rather involved, however, and it would not be practical to reproduce the necessary steps in this discussion.

From the relation expressed by (62) it is seen that the solution of the entire problem depends in no way upon the absolute values of the densities but only upon their ratio $\dfrac{\rho'}{\rho}$. Taking a figure of 0.960,788 for this ratio, α, κ^1, and κ^2 have the following values:

$$\alpha=0.980,198$$

$$\kappa_1=\frac{0.710,634}{\lambda}$$

$$\kappa_2=\frac{7.106,335}{\lambda}$$

This particular choice of $\dfrac{\rho'}{\rho}$ makes κ_2 exactly ten times larger than κ_1 which simplifies the calculation slightly.

Corresponding to these conditions the eight constants to be determined assume the numerical values given below in terms of λ. It will be found that these figures satisfy the last eight equations sufficiently closely for practical purposes.

$$c = +0.336,10\lambda$$

$$a = +0.159,07\lambda$$

$$Q^{I} = +0.009,08\lambda$$

$$Q^{III} = +0.010,78\lambda$$

$$K_1 = -0.769,48\lambda$$

$$K_2 = +0.150,86\lambda$$

$$Q_1 = +0.622,61\lambda$$

$$Q_2 = +0.014,20\lambda$$

Inserting these figures in the expression for the depths discussed previously, and assuming a value of D_0 equal

right of the free surface and the generation of mild easterly winds as shown in the velocity diagram. In region III there is also a shift of the fluid to the right amounting to 0.00771λ at $y = -0.3361\lambda$ and decreasing to zero for large negative values of y. This causes a piling up of the fluid in this region so that the free surface again slopes downward to the right and provides the necessary Coriolian pressure gradient for the mild easterly winds indicated on the velocity chart. In region II the original discontinuity M is located at $y = -0.15907\lambda$. In the lower layer there exist easterly winds which at the boundary $y = 0$ are equal in intensity to those in region I at this point, since continuity of displacement exists at the boundary. Farther to the right these increase rapidly and reach a maximum at the point $y = -0.3361\lambda$. The Coriolian pressure gradient, as already mentioned, is here provided by the inclination of the internal boundary and of the free surface.

In the upper layer the lateral displacement of the vertical columns may be thought of as the sum of two effects.
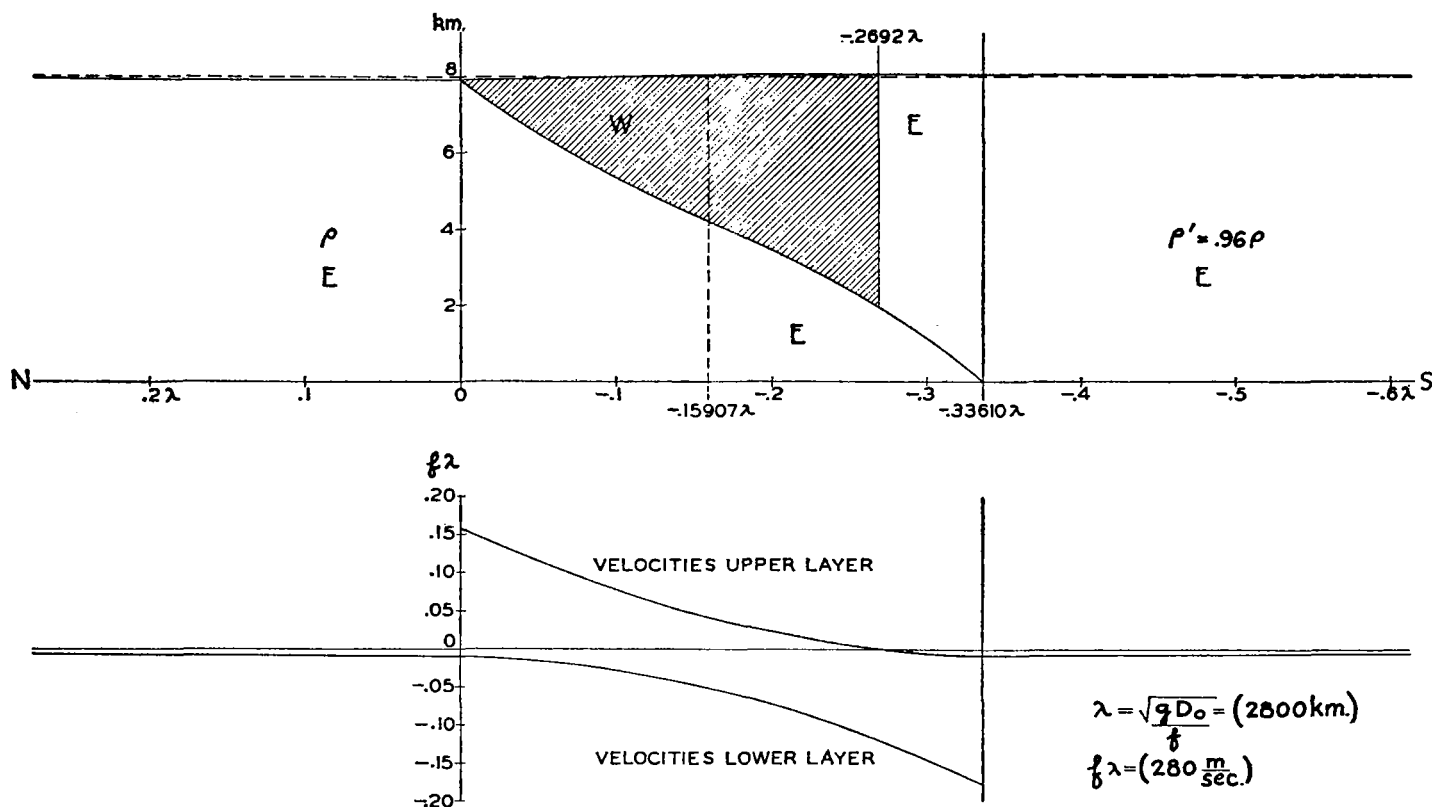
FIGURE 3.—Readjustment of two air masses under conservation of vorticity.

to the height of the homogeneous atmosphere which is taken to be eight kilometers, the shape of the free surface and of the internal boundary may be plotted. In this way the diagram reproduced in figure 3 is obtained. In middle latitudes the value of λ is approximately 2,800 km., which gives a horizontal scale for the diagram.

Likewise the velocities may now be obtained. These are also plotted in figure 3, the unit of velocity being $f\lambda$ which in middle latitudes has the approximate value of 280 meters per second. The directions are taken as corresponding to the Northern Hemisphere.

The general features of the final state may now be described as follows: Beginning with region I, there is here a shift of the fluid to the right amounting to 0.00908λ at the point $y = 0$, and decreasing with increasing positive values of y. This leads to a slope downward to the

In the first place, since there is a shift of the fluid in region III to the right, and since there must be continuity of displacement at the vertical boundary, the liquid in the upper layer in the vicinity of the boundary must take part in this motion to the right. Secondly, the vertical shrinking of the fluid columns from the original height D_0 to that shown on figure 3 must lead to a displacement to the left in order to preserve continuity of volume. Because these two effects oppose one another, there must exist a vertical in the upper layer of region II at which the displacement, the velocity, pressure gradient and slope of the free surface are all zero. By setting $y_0^{II'} = y$ in equation (63) this vertical is found to be at $y = -0.2692\lambda$. It is also apparent that the elevation of the free surface must reach a maximum at this point. This turns out to be 8.064 km., or 64 meters

above the initial depth of the fluid layer. The velocities for the upper layer are as shown, being mild easterly to the right of this vertical and increasing westerly to the left. The free surface has an increasing slope downward to the left, the depth reaching a minimum at $y=0$ where the value is 7.927 km. or 73 meters below the initial depth.

Having obtained a description of the equilibrium state, it becomes a matter of interest to examine the energy changes that must occur during the transition. Initially the system possesses no kinetic energy, but has a certain amount of potential energy of mass distribution. In the final state a certain kinetic energy of transverse circulation exists, while the potential energy is less than originally. It is now proposed to make a quantitative calculation of these energy changes.

The change of potential energy between the two states in region I may be expressed in the form of the following integral; energy rendered available being reckoned as positive:

$$\int_0^\infty \left(\frac{D_0}{2} - \frac{D^I}{2}\right) \rho g D^I dy. \tag{85}$$

Upon substitution for D^I from (14) and integration between the limits indicated it is found that this quantity assumes the value of 0.004,519 $\rho g D_0{}^2 \lambda$. Taking cognizance of the fact that one dimension of length is suppressed, it will be seen that this quantity has the dimensions of energy.

The corresponding integral for region III gives the result that the change in potential energy is $-0.003,866$ $\rho' g D_0{}^2 \lambda$ or $-0.003,714 \rho g D_0{}^2 \lambda$.

In region II it is convenient to calculate the change in potential energy for each layer independently. In the lower layer an integral of the form of (85) may be applied directly, although the integration is more laborious than in regions I and III. The result is that the change in potential energy is equal to 0.030,442$\rho g D_0{}^2 \lambda$. For the upper layer the integral has the form

$$\int_{-0.3361\lambda}^0 \left[\frac{D_0}{2} - \left(D^{II} + \frac{D^{II'}}{2}\right)\right] \rho' g D^{II'} dy. \tag{86}$$

The evaluation of this quantity gives as a result $-0.031,174\rho' g D_0{}^2 \lambda$ or $-0.029,952 \rho g D_0{}^2 \lambda$.

Summing for the three regions the net change in potential energy turns out to be $+0.001,295\rho g D_0{}^2 \lambda$.

The kinetic energy of the transverse circulation for region I may be expressed in the form of the integral,

$$\int_0^\infty \frac{u^{I2} \rho D^I dy}{2}, \tag{87}$$

which turns out to have a value of 0.000,020 $\rho f^2 D_0 \lambda^3$. The unit of energy appearing here will be seen to be equivalent to that occurring in the calculation of potential energy if the value of $\frac{g D_0}{f^2}$ be substituted for λ^2. Corresponding integrals may be set up for the remaining regions, giving the result that the kinetic energy is 0.000,150$\rho f^2 D_0 \lambda^3$ and 0.000,257$\rho f^2 D_0 \lambda^3$ for the upper and lower layers of region II, respectively, and 0.000,029 $\rho f^2 D_0 \lambda^3$ for region III. Summing for the three regions, the total kinetic energy of transverse circulation for the system becomes 0.000,456$\rho f^2 D_0 \lambda^3$.

Comparing this figure with that for the potential energy liberated during the readjustment, it is seen that the latter

is about three times as large as the former, and the question arises as to the physical meaning of the large difference between these quantities. It will be recalled that in the original statement of the problem the assumption was made that the transition from the initial to the final state proceeds infinitely slowly. Under these conditions it is permissible to neglect all accelerations in the y direction so that the system arrives at the equilibrium state without any finite velocities in the y direction.

Had this restriction not been made, but instead the readjustment allowed to take place at its natural rate, the solution obtained above would no longer describe completely the conditions when the system arrives at the equilibrium state. In addition to the transverse velocities in the x direction there would also exist finite velocities in the y direction. By virtue of these velocities the readjustment would proceed beyond the equilibrium state, then reverse its direction and the system would continue to oscillate about the equilibrium position. It appears that the fraction of the potential energy liberated which is not used to establish a transverse circulation represents energy of inertia oscillations whose character, however, cannot be in any way determined from this investigation.

It remains to be pointed out, however, that in general differences in density in the atmosphere are set up largely by gradual radiative processes, and that readjustment of actual atmospheric systems takes place simultaneously with the slow establishment of density differences. For this reason readjustment processes must automatically take place on a nearly quasistatic basis, passing through an infinite number of equilibrium stages, and arriving at the final state without appreciable energy of oscillation. Under these conditions the amount of heat energy required to bring about the change in density is less than would be required if the heating and expansion took place prior to the readjustment of mass distribution.

As a variation of the above problem it would be of interest to determine the changes that would be produced if a third mass of relatively lighter fluid overlies the two overturning air masses. Strictly speaking, since there would be upon readjustment a deformation of the internal boundary between the two layers thus formed, lateral convergence and divergence accompanied by the generation of weak transverse circulation and also a deformation of the free surface should be expected in the upper layer. If, however, the upper layer is relatively deep, these effects would be sufficiently small to be neglected, the upper layer being considered at rest with a level surface in the final state as well as initially.

An examination of the conditions imposed by the addition of such a layer shows that almost the same fundamental equations as were previously derived will serve to specify the solution of a problem of this type, provided that for each set of densities chosen, the initial depths of the two air masses comprising the lower layer bear a given fixed ratio to each other. We may thus represent the initial (dashed lines) and final states schematically as in figure 4.

In region I the pressure at the surface may be expressed as follows:

$$p_0 = \rho D^I + \rho''(D_0' - D^I) + p_c \tag{88}$$

where p_0 is the pressure at the level D_0'. When this is differentiated with respect to y,

$$\frac{\partial p_0}{\partial y} = (\rho - \rho'')\frac{\partial D^I}{\partial y}. \tag{89}$$

$$\rho''$$

FIGURE 4.—Schematic representation of final state of system containing three air masses.

Incorporating this expression for the pressure gradient in equation (6) and proceeding as in the previous problem, the fundamental equation for this region is found to be,

$$y_0{}^{\mathrm{I}}=y+Q^{\mathrm{I}}e^{-\frac{y}{\lambda^{\mathrm{I}}}} \tag{90}$$

in which λ^{I} is now defined as

$$\lambda^{\mathrm{I}}=\frac{\sqrt{gD_0\,\dfrac{\rho-\rho''}{\rho}}}{f}. \tag{91}$$

Exactly similar considerations lead to the following fundamental equation for region III,

$$y_0{}^{\mathrm{III}}=y+Q^{\mathrm{III}}e^{\frac{y}{\lambda^{\mathrm{III}}}} \tag{92}$$

where λ^{III} is given by

$$\lambda^{\mathrm{III}}=\frac{\sqrt{gD_0'\,\dfrac{\rho'-\rho''}{\rho'}}}{f} \tag{93}$$

In region II the pressure gradients may be obtained from the following expressions for $p_a{}'$ and p_a:

$$p_a{}'=g\rho'(D^{\mathrm{II}}+D^{\mathrm{II}'}-Z)+g\rho''(D_0'-D^{\mathrm{II}}-D^{\mathrm{II}'})+p_c, \tag{94}$$

$$p_a=g\rho D^{\mathrm{II}}+g\rho'D^{\mathrm{II}'}+g\rho''(D_0'-D^{\mathrm{II}}-D^{\mathrm{II}'})+p_c. \tag{95}$$

Differentiating these

$$\frac{\partial p_a{}'}{\partial y}=g\rho'\left(\frac{\partial D^{\mathrm{II}}}{\partial y}+\frac{\partial D^{\mathrm{II}'}}{\partial y}\right)-g\rho''\left(\frac{\partial D^{\mathrm{II}}}{\partial y}+\frac{\partial D^{\mathrm{II}'}}{\partial y}\right); \tag{96}$$

$$\frac{\partial p_a}{\partial y}=g\rho\frac{\partial D^{\mathrm{II}}}{\partial y}+g\rho'\frac{\partial D^{\mathrm{II}'}}{\partial y}-g\rho''\frac{\partial D^{\mathrm{II}'}}{\partial y}. \tag{97}$$

Carrying out similar operations as previously, an equation comparable to (40) is obtained:

$$0=f^2[y_0{}^{\mathrm{II}}+\alpha y_0{}^{\mathrm{II}'}-(1+\alpha)y]-gD_0\left[\frac{\rho-\rho''}{\rho}\right.$$
$$\left.+\frac{\rho'-\rho''}{\rho'}\right]\frac{d^2}{dy^2}\left[y_0{}^{\mathrm{II}}+\frac{D_0'\left(\dfrac{\rho'-\rho''}{\rho}+\dfrac{\rho'-\rho''}{\rho'}\alpha\right)}{D_0\left(\dfrac{\rho-\rho''}{\rho}+\dfrac{\rho'-\rho''}{\rho'}\alpha\right)}y_0{}^{\mathrm{II}'}\right] \tag{98}$$

Again defining α by equating the coefficients of $y_0{}^{\mathrm{II}'}$ and solving, the following equation results:

$$\alpha=\frac{1}{2}\left[\frac{D_0'}{D_0}-\frac{\rho'(\rho-\rho'')}{\rho(\rho'-\rho'')}\pm\sqrt{\left[\frac{D_0'}{D_0}-\frac{\rho'(\rho-\rho'')}{\rho(\rho'-\rho'')}\right]^2+\frac{4\rho'D_0'}{\rho D_0}}\right] \tag{99}$$

If now the fundamental equations for region II are to have the same form as (63) and (64), it is necessary that the roots of this equation differ only as to sign, as was the case previously. This imposes the condition that

$$\frac{D_0'}{D_0}-\frac{\rho'(\rho-\rho'')}{\rho(\rho'-\rho'')}=0. \tag{100}$$

Retaining the same ratio of densities in the lower layer as was used in the first problem and assuming that $\rho''=0.80\rho$, the value of the ratio $\dfrac{D_0'}{D_0}$ turns out to be 1.195,098.

The quantity α now becomes equal to $\sqrt{\dfrac{\rho'D_0'}{\rho D_0}}$, while the constants κ_1, κ_2, and λ are defined as follows:

$$\kappa_1=\frac{1}{\lambda\sqrt{\dfrac{\rho-\rho''}{\rho}+\dfrac{\rho'-\rho''}{\rho'}\alpha}} \tag{101}$$

$$\kappa_2=\frac{1}{\lambda\sqrt{\dfrac{\rho-\rho''}{\rho}-\dfrac{\rho'-\rho''}{\rho'}\alpha}} \tag{102}$$

$$\lambda=\frac{\sqrt{gD_0}}{f} \tag{103}$$

The condition (100) also causes λ^{I} to be equal to λ^{III}.

The determination of the constants to be fixed by boundary conditions can be done in almost exactly the same manner as before, leading to the numerical values given below:

$$c = +0.338,15\lambda$$
$$a = +0.180,20\lambda$$
$$Q^{I} = -0.007,12\lambda$$
$$Q^{III} = -0.027,72\lambda$$
$$K_1 = -0.425,19\lambda$$
$$K_2 = +0.171,17\lambda$$
$$Q_1 = +0.224,97\lambda$$
$$Q_2 = +0.014,81\lambda$$

The expressions for the depths and for the velocities are formed as previously, and if the results are plotted on the

SUMMARY

1. If, on a rotating earth, a layer of incompressible fluid of uniform depth and infinite horizontal extent composed of two motionless fluid masses of differing density and separated initially by a plane vertical partition, is allowed to readjust itself slowly upon removal of the partition, the system will come to a state of equilibrium in which the two masses are separated by an inclined boundary sloping downward toward the lighter fluid mass, and in which the free surface is deformed in such a manner that the maximum depth occurs at a point above the internal boundary and the minimum depth at the point where the thickness of the lighter fluid becomes zero. In obtaining this result and those that follow, all frictional effects as well as mixing at the mutual boundary of the fluids, are assumed to be absent. The Coriolis parameter is not assumed to vary within the dimensions of the system. For an initial
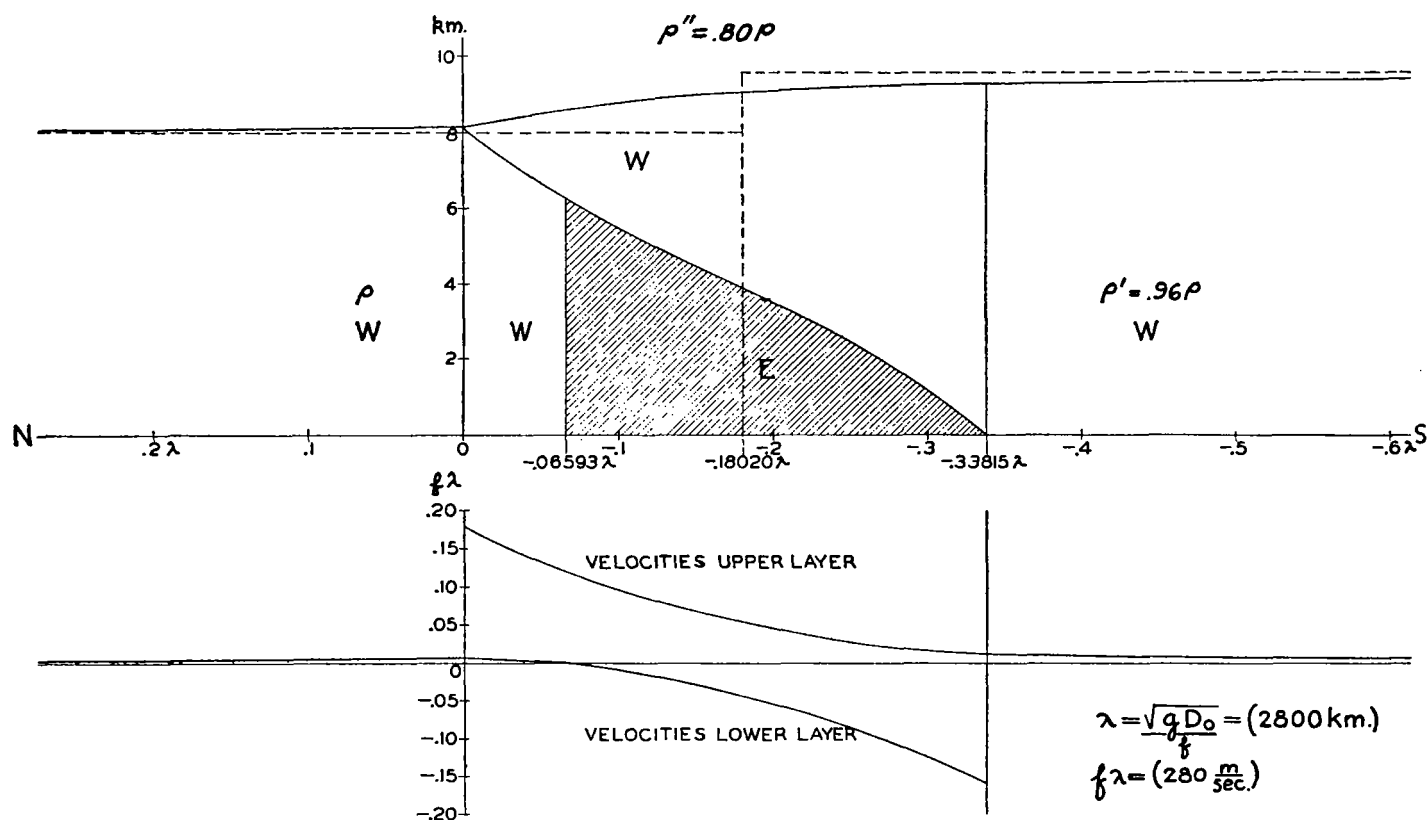
FIGURE 5.—Readjustment of three air masses under conservation of vorticity.

same scale as before the diagram shown in figure 5 is obtained. It is seen that now a weak west wind prevails in regions I and III and also in the left portion of the lower layer of region II. The upper layer now has a continuous distribution of westerly winds, while the right hand portion of the lower layer in region II has easterly winds with velocities as shown. The initial state is shown by the dashed lines.

In concluding, it might be remarked that the above analyses depend essentially upon the fact that in an incompressible fluid of uniform density, that is, in a homogeneous and incompressible fluid, there can be no variation of gradient wind with elevation. This is also true for a homogeneous and compressible fluid such as an adiabatic atmosphere. It should therefore be possible, at least in theory, to carry out a similar calculation for a system composed of two air masses each having a constant potential temperature.

depth of 8 km. and a ratio of densities equal to 0.96 the shape of the free surface and internal boundary is given by figure 3.

2. Due to the action of pressure gradients and Coriolis forces during the readjustment of the above system, horizontal velocities parallel to the original discontinuity will be set up. The distribution of these velocities will be such that if in the Northern Hemisphere the discontinuity is imagined, for the purpose of reference, as extending from east to west with the lighter fluid to the south, easterly velocities will appear in the denser fluid, decreasing to zero at great distances to the north and increasing southward to a maximum at the point where the thickness of the heavier fluid becomes zero.

In the lighter fluid the velocity will be zero at great distances to the south and also along the vertical at which the elevation of the free surface is a maximum. To the south of this vertical the velocities will be mild easterly,

having a maximum above the point where the internal boundary intersects the lower surface of the layer. To the north of this vertical, westerly velocities increasing northward will prevail, reaching a maximum at the point where the thickness of the lighter fluid becomes zero. Within each mass of fluid there will be no variation of velocity with elevation. A graph of the velocities for a density ratio of 0.96 and an initial depth of 8 km. is given in figure 3. The unit of velocity used is a function of geographical latitude and amounts to about 280 meters per second in middle latitudes.

3. In the above system there will be a decrease of potential energy of mass distribution during the readjustment. Also, there will exist in the final state a certain amount of kinetic energy represented by the velocity distribution already described. Computation on the example already cited shows that the amount of potential energy set free is nearly three times as great as the kinetic energy represented by the circulaiton set up. The difference between these two quantities is accounted for by the fact that the readjustment was assumed to take place slowly on a quasistatic basis, implying the existence of an external retarding agency which absorbs a portion of the potential energy rendered available.

If the process is considered as taking place at its natural rate, the system will then arrive at the equilibrium point with a certain amount of kinetic energy of oscillation. If this additional kinetic energy is included, the sum of potential and kinetic energy for the system remains constant at all stages of the process. The character of the oscillations performed by the system cannot be determined from this analysis. However, it is probable that in the actual atmosphere density differences are generally set up gradually, so that readjustment processes automatically proceed in an almost quasistatic manner, and energy of oscillation does not appear to a significant extent.

4. A problem comparable to the above but involving the presence of a third mass of fluid overlying the system can be treated by almost exactly the same mathematical set-up save for numerical values as was used in the first analysis, provided that the depth of the third mass is relatively large in comparison with the other two, and that the initial depths of the first two masses bear the following relation to the densities chosen:

$$\frac{D_0'}{D_0} = \frac{\rho'(\rho - \rho'')}{\rho(\rho' - \rho'')},$$

where D_0 and D_0' are the depths of the heavier and lighter fluids respectively and ρ and ρ' their densities, while ρ'' is the density of the third mass.

Taking the same value for the ratio of densities in the lower two masses and assuming a value of ρ'' equal to 0.80 ρ, and a value of 8 km. for D_0, the result shown in figure 5 is obtained. The diagram is to be interpreted in the same manner as figure 3.

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